A non-linear Oscillator with quasi-Harmonic behaviour: two- and n-dimensional Oscillators

José F. Cariñena† a), Manuel F. Rañada† b), Mariano Santander‡ c) and Murugaian Senthilvelan‡ d)

- † Departamento de Física Teórica, Facultad de Ciencias Universidad de Zaragoza, 50009 Zaragoza, Spain
- ‡ Departamento de Física Teórica, Facultad de Ciencias Universidad de Valladolid, 47011 Valladolid, Spain
- # Centre for Nonlinear Dynamics, Bharathidasan University Tiruchirapalli 620 024, India

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Abstract

A nonlinear two-dimensional system is studied by making use of both the Lagrangian and the Hamiltonian formalisms. The present model is obtained as a two-dimensional version of a one-dimensional oscillator previously studied at the classical and also at the quantum level. First, it is proved that it is a super-integrable system, and then the nonlinear equations are solved and the solutions are explicitly obtained. All the bounded motions are quasiperiodic oscillations and the unbounded (scattering) motions are represented by hyperbolic functions. In the second part the system is generalized to the case of n degrees of freedom. Finally, the relation of this nonlinear system with the harmonic oscillator on spaces of constant curvature, two-dimensional sphere S^2 and hyperbolic plane H^2 , is discussed.

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 $^{^{}a)}E$ -mail address: jfc@unizar.es

b) E-mail address: mfran@unizar.es

c) E-mail address: santander@fta.uva.es

d) E-mail address: senthilvelan@cnld.bdu.ac.in

1 Introduction and main results

Mathews and Lakshmanan studied in 1974 [1],[2], the equation

$$(1 + \lambda x^2) \ddot{x} - (\lambda x) \dot{x}^2 + \alpha^2 x = 0, \quad \lambda > 0,$$
 (1)

as an example of a non-linear oscillator (notice α^2 was written just as α in the original paper). In fact they considered (1) as a particular case of the differential equation

$$y'' + f(y)y'^{2} + g(y) = 0,$$

that can be solved by using a two steps procedure: (i) first a reduction of order can be fulfilled by the change $y'=p,\ y''=pp',\ p'=dp/dy;$ (ii) then the corresponding first order equation can be solved by using $\mu(y)=\exp\{2\int f(y)\,dy\}$, as an integrating factor. In this case we have $f(x)=-\lambda\,x/(1+\lambda x^2)$ and hence $\mu(x)=1/(1+\lambda\,x^2)$; the general solution takes the form

$$x = A\sin(\omega t + \phi)$$
,

with the following additional restriction linking frequency and amplitude

$$\omega^2 = \frac{\alpha^2}{1 + \lambda A^2} \,.$$

That is, the equation (1) represents a non-linear oscillator with periodic solutions that they qualify as having a "simple harmonic form". The authors also proved that (1) is obtainable from the Lagrangian

$$L = \frac{1}{2} \left(\frac{1}{1 + \lambda r^2} \right) (\dot{x}^2 - \alpha^2 x^2) \tag{2}$$

which they considered as the one-dimensional analogue of the Lagrangian density

$$L = \frac{1}{2} \left(\frac{1}{1 + \lambda \phi^2} \right) \left(\partial_{\mu} \phi \, \partial^{\mu} \phi - m^2 \, \phi^2 \right), \tag{3}$$

appearing in some models of quantum field theory [3],[4].

The equation (1) is therefore an interesting example of a system with nonlinear oscillations with a frequency (or period) showing amplitude dependence. As a quantum system, the one-dimensional Schroedinger equation involving the potential $x^2/(1+gx^2)$ was considered in [5] as an example of anharmonic oscillator, and later on studied in [6]-[10]; the three-dimensional quantum problem was considered in [11],[12] (e.g., in [11] the Bohr-Sommerfeld quantization procedure was applied in relation with some previous studies [13],[14], in nonpolynomial quantum mechanical models). We observe that this system can also be considered as an oscillator with a position-dependent effective mass (see [15] and references therein).

The main objective of this article is to develop a deeper analysis of the equation (1) and the Lagrangian (2), first proving that this particular λ -dependent nonlinear system can be generalized to the two-dimensional case, and even to the n-dimensional case, and second,

pointing towards a simple geometric interpretation of the system so obtained. In more detail, the plan of the article is as follows: Sec. 2 is devoted to the properties of the λ -dependent kinetic part of the Lagrangian and the λ -dependent two-dimensional free motion. In Sec. 3, that must be considered as the central part of this article, we study the λ -dependent two-dimensional oscillator; we have divided this section in three parts: in the first part we discuss the existence of Noether symmetries for λ -dependent Lagrangians of the form

$$L(x, y, v_x, v_y; \lambda) = \frac{1}{2} \left(\frac{1}{1 + \lambda r^2} \right) \left[v_x^2 + v_y^2 + \lambda (xv_y - yv_x)^2 \right] - V(x, y; \lambda).$$

In the second part we discuss the properties of the λ -oscillator described by the Lagrangian

$$L = \frac{1}{2} \left(\frac{1}{1 + \lambda r^2} \right) \left[v_x^2 + v_y^2 + \lambda \left(x v_y - y v_x \right)^2 \right] - \frac{\alpha^2}{2} \left(\frac{r^2}{1 + \lambda r^2} \right), \quad r^2 = x^2 + y^2,$$

first proving that such system is superintegrable and then solving the equations of motion using the Lagrangian formalism as an approach; the third part deals with the Hamiltonian formalism and the λ -dependent Hamilton-Jacobi equation which is shown to be separable in three different coordinate systems. In Sec. 4 we study the λ -dependent n-dimensional nonlinear oscillator described by the Lagrangian

$$L = \frac{1}{2} \left(\frac{1}{1 + \lambda r^2} \right) \left[\sum_{i} v_i^2 + \lambda \sum_{i \le j} J_{ij}^2 \right] - \frac{\alpha^2}{2} \left(\frac{r^2}{1 + \lambda r^2} \right), \quad r^2 = \sum_{i} x_i^2,$$

and we solve the associated equations obtaining different types of solutions depending of the values of λ . Next we consider the Hamiltonian approach, we prove that it is a superintegrable system and we obtain different families of constants of motion; the existence of several different sets of n commuting integrals is also discussed. In Sec. 5 we start with a discussion of Lie algebra structure of the symmetries of the n=2 non-linear oscillator; in fact we see that they span a three-dimensional real Lie algebra isomorphic to $SO(3,\mathbb{R})$, SO(2,1) or the Euclidean group in two dimensions, depending on the sign of the parameter λ . Then we present a geometric approach which explains the surprising properties this λ -dependent system has and relates it to the harmonic oscillator in spaces of constant curvature studied in Ref. [16]-[18]. Finally, in Sec. 6 we make some final comments.

2 λ -dependent "Free Particle"

We will make use of the Lagrangian formalism as an approach. That is, we will look, in the first place, for a Lagrangian function $L(\lambda)$ with appropriate properties and then, we will turn our attention to the corresponding nonlinear equations arising from $L(\lambda)$.

It is clear that the equation (1) represents a non-linear version of a linear equation with a non-linearity introduced by the coefficient λ ; but the important point is that, in Lagrangian terms, this coefficient λ modifies not only the quadratic potential $V = (1/2) x^2$ of the

harmonic oscillator but also the kinetic term $T = (1/2) v_x^2$. Therefore, this particular system is not directly related with Henon-Heiles or any other similar non-linear system [19]-[22] where the nonlinearity is introduced by just adding a new term of higher order to the original potential. Our first aim is to extend to n = 2 dimensions this system in such a way that its distinguishing properties are maintained. So in order to construct the appropriate two-dimensional Lagrangian, we may split the problem in two: the problem of the kinetic term and the problem of the potential.

Next we begin with that of the kinetic term.

Before giving the expression for the two–dimensional kinetic term, let us consider some properties of the one-dimensional free-particle motion characterized by the following Lagrangian

$$L(x, v_x; \lambda) = T_1(\lambda) = \frac{1}{2} \left(\frac{v_x^2}{1 + \lambda r^2} \right), \tag{4}$$

and the following equation

$$(1 + \lambda x^2) \ddot{x} - (\lambda x) \dot{x}^2 = 0, \quad \lambda > 0.$$
 (5)

Two important properties are: (i) The function $T_1(\lambda)$ is invariant under the action of the vector field $X_x = X_x(\lambda)$ given by

$$X_x(\lambda) = \sqrt{1 + \lambda x^2} \frac{\partial}{\partial x},$$

in the sense that we have

$$X_x^t(\lambda)(T_1(\lambda)) = 0$$
,

where $X_x^t(\lambda)$ denotes the natural lift to the phase space $\mathbb{R} \times \mathbb{R}$ (tangent bundle in differential geometric terms) of the vector field $X_x(\lambda)$,

$$X_x^t(\lambda) = \sqrt{1 + \lambda x^2} \frac{\partial}{\partial x} + \left(\frac{\lambda x v_x}{\sqrt{1 + \lambda x^2}}\right) \frac{\partial}{\partial v_x}.$$

(ii) The general solution of the equation of motion, that can be directly obtained from the conservation of the energy E, is given by

$$x = \left(\frac{1}{\sqrt{\lambda}}\right) \sinh\left(\sqrt{2\lambda E} \left(t + \phi\right)\right). \tag{6}$$

It is clear that this expression satisfies correctly the linear limit for $\lambda = 0$.

The first problem of the transition from 1-d to 2-d configuration space is the construction of the new two-dimensional $T_2(\lambda)$. Many different λ -dependent functions will have the same $\lambda = 0$ limit, so we must require that the new function $T_2(\lambda)$ must satisfy certain properties. On the one hand we think that a natural requirement for $T_2(\lambda)$ is to satisfy the two-dimensional versions of the previous points (i) and (ii). On the other hand the potential $V(\lambda)$ for the two-dimensional oscillator (to be studied in the next section) must be a λ -dependent central

potential such that the angular momentum be preserved; but, according to the Noether theorem, exact symmetries of the potential lead to constants of motion only if they also are symmetries of the "kinetic energy". Hence, a necessary condition must be that $T_2(\lambda)$ be also preserved with the same symmetry.

We will consider as the starting point for our approach the following three requirements.

- 1. The kinetic term $T_2(\lambda)$ must be a quadratic function of the velocities that will remain invariant under rotations in the \mathbb{R}^2 plane. This means that it must depend of the coordinates x, y, by means of $r^2 = x^2 + y^2$, and of the velocities v_x , v_y , by means of $v_x^2 + v_y^2$, $(xv_y yv_x)^2$, $(xv_x + yv_y)^2$ and $(xv_x + yv_y)(xv_y yv_x)$.
- 2. $T_2(\lambda)$ should be invariant under (the lifts of) the two vector fields $X_1(\lambda)$ and $X_2(\lambda)$ given by

$$X_1(\lambda) = \sqrt{1 + \lambda r^2} \frac{\partial}{\partial x},$$

 $X_2(\lambda) = \sqrt{1 + \lambda r^2} \frac{\partial}{\partial y},$

that represent the natural extension to \mathbb{R}^2 of the vector field $X_x(\lambda)$ associated to $T_1(\lambda)$ in the one-dimensional case. Notice that the previous point (1) implies that if $T_2(\lambda)$ is invariant under (the lift of) $X_1(\lambda)$ then so will be under (the lift of) $X_2(\lambda)$.

3. It must lead to a solution (x(t), y(t)) in \mathbb{R}^2 for the two-dimensional free-particle motion similar to the hyperbolic function x(t) in \mathbb{R} given by (6), for the one-dimensional free-particle motion.

The most direct and simplest generalization of the one-dimensional kinetic term $T_1(\lambda)$ is given by

$$T(x, y, v_x, v_y; \lambda) = (\frac{1}{2}) \left(\frac{v_x^2 + v_y^2}{1 + \lambda (x^2 + y^2)} \right),$$

but this function does not satisfy the point (2). So let us try a more general expression given by

$$T(x, y, v_x, v_y; \lambda) = (\frac{1}{2}) \frac{W(v_x, v_y)}{1 + \lambda r^2},$$

where $W = W(v_x, v_y)$ denotes

$$W = c_1(v_x^2 + v_y^2) + c_2(xv_y - yv_x)^2 + c_3(xv_x + yv_y)^2 + c_4(xv_x + yv_y)(xv_y - yv_x),$$

then the point (2) is satisfied only if

$$c_2 = \lambda c_1, \quad c_3 = 0, \quad c_4 = 0.$$

Hence, we will choose the following two-dimensional kinetic function

$$T_2(\lambda) = \left(\frac{1}{2}\right) \left(\frac{1}{1+\lambda r^2}\right) \left[v_x^2 + v_y^2 + \lambda \left(xv_y - yv_x\right)^2\right], \quad r^2 = x^2 + y^2, \tag{7}$$

as the appropriate one for the two-dimensional λ -dependent dynamics. Notice that this means that the λ -dependence is introduced in two different ways: the first one is the global factor $1/(1 + \lambda r^2)$ that is the most direct n = 2 extension of the one-dimensional factor $1/(1 + \lambda x^2)$ in (2); the other λ -term is not so simple and it represents a two-dimensional contribution that was not present (in fact, it can not be defined) in the one-dimensional case. Although one could guess that this additional term can introduce difficulties, we will see that it really simplifies most of properties, mainly all those related with symmetries (the geometric aspects will be discussed in Sec. 5). We also point out that we admit λ can take both positive and negative values. It is clear that for $\lambda < 0$, $\lambda = -|\lambda|$, the function (and the associated dynamics) will have a singularity at $1 - |\lambda| r^2 = 0$; because of this we will restrict the study of the dynamics to the interior of the circle $x^2 + y^2 < 1/|\lambda|$ that is the region in which $T_2(\lambda)$ is positive definite.

It is known that a symmetric bilinear form in the velocities (v_x, v_y) can be considered as associated to a two-dimensional metric ds^2 in \mathbb{R}^2 . In this particular case, the function $T_2(\lambda)$ considered as a bilinear form determines the following λ -dependent metric

$$ds^{2}(\lambda) = \left(\frac{1}{1+\lambda r^{2}}\right) \left[(1+\lambda y^{2}) dx^{2} + (1+\lambda x^{2}) dy^{2} - 2\lambda xy dx dy \right].$$
 (8)

This relation between kinetic term and metric implies that the Killing vectors of the λ -metric $ds^2(\lambda)$ coincide with the exact Noether symmetries of the λ -dependent free motion, that is, of the dynamics determined by assuming the kinetic term as Lagrangian, $L(\lambda) = T_2(\lambda)$.

 $T_2(\lambda)$ remains invariant under the actions of the lifts of the vector fields $X_1(\lambda)$, $X_2(\lambda)$, and X_J , given by

$$X_{1}(\lambda) = \sqrt{1 + \lambda r^{2}} \frac{\partial}{\partial x},$$

$$X_{2}(\lambda) = \sqrt{1 + \lambda r^{2}} \frac{\partial}{\partial y},$$

$$X_{J} = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x},$$

in the sense that, if we denote by X_r^t , r=1,2,J, the natural lift to the tangent bundle (phase space $\mathbb{R}^2 \times \mathbb{R}^2$) of the vector field X_r ,

$$\begin{array}{lll} X_1^t(\lambda) & = & \sqrt{1+\lambda\,r^2} \,\, \frac{\partial}{\partial x} + \lambda\, \Big(\frac{xv_x+yv_y}{\sqrt{1+\lambda\,r^2}}\Big) \frac{\partial}{\partial v_x}\,, \\ X_2^t(\lambda) & = & \sqrt{1+\lambda\,r^2} \,\, \frac{\partial}{\partial y} + \lambda\, \Big(\frac{xv_x+yv_y}{\sqrt{1+\lambda\,r^2}}\Big) \frac{\partial}{\partial v_y}\,, \end{array}$$

then the Lie derivatives of $T_2(\lambda)$ with respect to $X_r^t(\lambda)$ vanish, that is

$$X_r^t(\lambda)(T_2(\lambda)) = 0$$
, $X_J^t(T_2(\lambda)) = 0$, $r = 1, 2$.

We close this section by solving the two-dimensional free-particle motion determined by two-dimensional kinetic function (7).

The Euler equations arising from $L(\lambda) = T_2(\lambda)$, and representing the n = 2 generalization of (5), are

$$(1 + \lambda r^{2}) \ddot{x} - \lambda [\dot{x}^{2} + \dot{y}^{2} + \lambda (x\dot{y} - y\dot{x})^{2}] x = 0,$$

$$(1 + \lambda r^{2}) \ddot{y} - \lambda [\dot{x}^{2} + \dot{y}^{2} + \lambda (x\dot{y} - y\dot{x})^{2}] y = 0.$$
(9)

Of course they are much more difficult of solving than the single equation (5); but assuming hyperbolic/trigonometric expressions for the two functions, x(t) and y(t), then we obtain that the solutions of (9) are given by

$$x = \left(\frac{A}{\sqrt{\lambda}}\right) \sinh\left(Ct + \phi_1\right), \quad y = \left(\frac{B}{\sqrt{\lambda}}\right) \sinh\left(Ct + \phi_2\right), \quad \lambda > 0,$$

$$x = \left(\frac{A}{\sqrt{|\lambda|}}\right) \sin\left(Ct + \phi_1\right), \quad y = \left(\frac{B}{\sqrt{|\lambda|}}\right) \sin\left(Ct + \phi_2\right), \quad \lambda < 0,$$
(10)

with the only restriction

$$A^2 + B^2 + A^2 B^2 \sinh^2(\phi_1 - \phi_2) = 1, \quad \lambda > 0,$$

 $A^2 + B^2 - A^2 B^2 \sin^2(\phi_1 - \phi_2) = 1, \quad \lambda < 0.$

After some calculus we arrive, for $\lambda > 0$ to

$$\lambda P_1^2 = C^2(1 - B^2), \quad \lambda P_2^2 = C^2(1 - A^2), \quad C^2 = 2\lambda E,$$

hence the solutions (x(t), y(t)) reduce to

$$x = \sqrt{\frac{P_1^2 - \lambda J^2}{2\lambda E}} \sinh\left(\sqrt{2\lambda E} (t + \phi_1)\right),$$

$$y = \sqrt{\frac{P_2^2 - \lambda J^2}{2\lambda E}} \sinh\left(\sqrt{2\lambda E} (t + \phi_2)\right),$$
(11)

for $\lambda > 0$, where J denotes the angular momentum $xv_y - yv_x$. For the case $\lambda < 0$ similar reasoning leads to

$$x = \sqrt{\frac{P_1^2 + |\lambda| J^2}{2|\lambda| E}} \sin\left(\sqrt{2|\lambda| E} (t + \phi_1)\right),$$

$$y = \sqrt{\frac{P_2^2 + |\lambda| J^2}{2|\lambda| E}} \sin\left(\sqrt{2|\lambda| E} (t + \phi_2)\right).$$
(12)

These results generalize the hyperbolic solution x(t) in \mathbb{R} given by (6) and satisfy correctly the linear limit for $\lambda = 0$.

3 λ -dependent n=2 quasi-Harmonic Oscillator

In this section we will study and solve the appropriate n=2 versions of the λ -dependent equation (1) and the λ -dependent Lagrangian (2).

3.1 Noether symmetries of λ -dependent potentials

A general standard λ -dependent Lagrangian (kinetic term minus a potential) will have the following form

$$L(x, y, v_x, v_y; \lambda) = T_2(\lambda) - V(x, y; \lambda)$$

in such a way that for $\lambda = 0$ we recover the non-deformed linear system.

It is known that if a potential V(x,y), defined the Euclidean plane, is invariant under either translations or rotations then it admits a Noether integral of first order in the velocities. Now, in this λ -dependent case, we have also a rather similar situation, but as the Lagrangian system is λ -dependent so are the transformations. The infinitesimal transformations generated by $X_1(\lambda)$ are

$$x' = x + \epsilon \, \delta x$$
, $y' = y$, $\delta x = \sqrt{1 + \lambda r^2}$,

and can be interpreted as a λ -dependent version of the translations along the x-axis. Similarly the generator of the one-parameter group of λ -dependent translations along the y-axis

$$x' = x$$
, $y' = y + \epsilon \delta y$, $\delta y = \sqrt{1 + \lambda r^2}$,

is the vector field $X_2(\lambda)$ (the generator of rotations remains λ -independent). If we denote by θ_L the Cartan semibasic one-form

$$\theta_L = \left(\frac{\partial L}{\partial v_x}\right) dx + \left(\frac{\partial L}{\partial v_y}\right) dy$$
$$= \left(\frac{1}{1 + \lambda r^2}\right) \left[v_x dx + v_y dy + \lambda (xv_y - yv_x)(x dy - y dx)\right],$$

then we have the following

1. If the potential $V(\lambda)$ does not depend on the variable x then the Lagrangian $L(\lambda)$ is invariant under the transformations generated by $X_1(\lambda)$; in this case the function $P_1(\lambda)$ given by

$$P_1(\lambda) = i(X_1^t(\lambda)) \theta_L = \frac{v_x - \lambda Jy}{\sqrt{1 + \lambda r^2}}$$

is a constant of motion. Notice that in this case the coordinate x is not cyclic since it is always present in kinetic term $T_2(\lambda)$.

2. If the potential $V(\lambda)$ is independent of the variable y then the Lagrangian $L(\lambda)$ is invariant under the transformations generated by $X_2(\lambda)$; in this case the function $P_2(\lambda)$ given by

$$P_2(\lambda) = i(X_2^t(\lambda)) \theta_L = \frac{v_y + \lambda Jx}{\sqrt{1 + \lambda r^2}}$$

is a constant of motion. This situation is similar to the previous one; that is, the coordinate y is not in the potential but, for $\lambda \neq 0$, it appears in the kinetic term.

3. If $V(\lambda)$ is a central potential, then

$$J = i(X_J^t) \, \theta_L = x v_y - y v_x$$

is a constant of motion. Notice that both the vector field X_J and J are λ -independent.

In these three very particular cases, the corresponding system becomes integrable with a second integral, $P_1(\lambda)$, $P_2(\lambda)$, or J, arising from an exact Noether symmetry.

3.2 Lagrangian approach

Let us consider the following λ -dependent Lagrangian

$$L = T_2(\lambda) - V_2(r; \lambda), \quad V_2(r; \lambda) = \frac{\alpha^2}{2} \left(\frac{r^2}{1 + \lambda r^2} \right), \quad r^2 = x^2 + y^2,$$
 (13)

where $V_2(r; \lambda)$ is the direct extension to n = 2 of the n = 1 potential in (2). The dynamics is given by the following λ -dependent vector field

$$\Gamma_{\lambda} = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + F_x(x, y, v_x, v_y; \lambda) \frac{\partial}{\partial v_x} + F_y(x, y, v_x, v_y; \lambda) \frac{\partial}{\partial v_y}$$

where the two functions F_x and F_y are given by

$$F_{x} = -\alpha^{2} \left(\frac{x}{1 + \lambda r^{2}} \right) + \lambda \left[v_{x}^{2} + v_{y}^{2} + \lambda J^{2} \right] \left(\frac{x}{1 + \lambda r^{2}} \right),$$

$$F_{y} = -\alpha^{2} \left(\frac{y}{1 + \lambda r^{2}} \right) + \lambda \left[v_{x}^{2} + v_{y}^{2} + \lambda J^{2} \right] \left(\frac{y}{1 + \lambda r^{2}} \right),$$

in such a way that for $\lambda = 0$ we recover the dynamics of the standard 2-d harmonic oscillator

$$\Gamma_0 = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} - (\alpha^2 x) \frac{\partial}{\partial v_x} - (\alpha^2 y) \frac{\partial}{\partial v_y}.$$

We see that for $\lambda < 0$ the potential $V_2(r; \lambda)$ is a well with a boundless wall at $r^2 = 1/|\lambda|$; therefore, all the trajectories will be bounded. For $\lambda > 0$ we have that $V_2(r; \lambda) \to (1/2)(\alpha^2/\lambda)$ for $r \to \infty$; so for small energies the trajectories will be bounded but for $E(\lambda) > (1/2)(\alpha^2/\lambda)$ the trajectories will be unbounded (see Figures I and II; notice that we have plotted $V(x, \lambda)$ but the graph of $V_2(r; \lambda)$ is just the same but with $r \ge 0$).

Our objective is to solve the λ -dependent equations arising from (13) and prove that this two-dimensional motion is periodic in the bounded case as it was in the one-dimensional case. At this point we recall that a system is called super-integrable if it is integrable (in the sense of Liouville-Arnold) and, in addition, possesses more independent first integrals than degrees of freedom; in particular, if a system with n degrees of freedom possesses N=2n-1 independent first integrals, then it is called maximally super-integrable. An important property is that the existence of periodic motions is a characteristic related with super-integrability; thus we may suspect that this system is super-integrable. This is actually the case, and the following proposition states the super-integrability of this λ -deformed system and proves the existence of a complex factorization.

Proposition 1 Let K_1 , K_2 , be the following two functions

$$K_1 = P_1(\lambda) + i \alpha \left(\frac{x}{\sqrt{1 + \lambda r^2}} \right),$$

$$K_2 = P_2(\lambda) + i \alpha \left(\frac{y}{\sqrt{1 + \lambda r^2}} \right).$$

Then the complex functions K_{ij} defined as

$$K_{ij} = K_i K_i^*, \quad i, j = 1, 2,$$

are constants of motion.

Proof: We begin our analysis by considering the action of the vector field Γ_{λ} , which represents the time-derivative, on the two λ -dependent functions $P_1(\lambda)$ and $P_2(\lambda)$. They are given by

$$\frac{d}{dt}P_1(\lambda) = -\left(\frac{\alpha^2}{1+\lambda r^2}\right)\left(\frac{x}{\sqrt{1+\lambda r^2}}\right), \qquad \frac{d}{dt}P_2(\lambda) = -\left(\frac{\alpha^2}{1+\lambda r^2}\right)\left(\frac{y}{\sqrt{1+\lambda r^2}}\right).$$

In a similar way, the time-derivative of the two velocity-independent functions, $x/\sqrt{1+\lambda r^2}$ and $y/\sqrt{1+\lambda r^2}$, is given by

$$\frac{d}{dt}\left(\frac{x}{\sqrt{1+\lambda r^2}}\right) = \left(\frac{1}{1+\lambda r^2}\right)P_1(\lambda), \qquad \frac{d}{dt}\left(\frac{y}{\sqrt{1+\lambda r^2}}\right) = \left(\frac{1}{1+\lambda r^2}\right)P_2(\lambda).$$

Thus, the time-evolution of the two functions, K_1 and K_2 , becomes

$$\frac{d}{dt} K_1 \equiv \Gamma_{\lambda}(K_1) = \frac{d}{dt} P_1(\lambda) + i \alpha \frac{d}{dt} \left(\frac{x}{\sqrt{1 + \lambda r^2}} \right)
= \left(\frac{1}{1 + \lambda r^2} \right) \left(i \alpha P_1(\lambda) - \frac{\alpha^2 x}{\sqrt{1 + \lambda r^2}} \right) = \left(\frac{i \alpha}{1 + \lambda r^2} \right) K_1,$$

and a similar calculus leads to

$$\frac{d}{dt} K_2 \equiv \Gamma_{\lambda}(K_2) = \left(\frac{i \alpha}{1 + \lambda r^2}\right) K_2.$$

Thus we obtain

$$\frac{d}{dt}(K_i K_j^*) \equiv \Gamma_{\lambda}(K_i K_j^*) = 0, \quad i, j = 1, 2,$$

Therefore the potential $V_2(\lambda)$ is super-integrable with the following three integrals of motion

$$I_1(\lambda) = |K_1|^2$$
, $I_2(\lambda) = |K_2|^2$, $I_3 = \operatorname{Im}(K_{12}) = \alpha (xv_y - yv_x)$.

That is, the existence of an invariant second order tensor K_{ij} , admitting a complex factorization [23, 24], is preserved by the nonlinearity introduced by λ .

The system of equations

$$(1 + \lambda r^2) \ddot{x} - \lambda [\dot{x}^2 + \dot{y}^2 + \lambda (x\dot{y} - y\dot{x})^2] x + \alpha^2 x = 0,$$

$$(1 + \lambda r^2) \ddot{y} - \lambda \left[\dot{x}^2 + \dot{y}^2 + \lambda (x\dot{y} - y\dot{x})^2 \right] y + \alpha^2 y = 0, \tag{14}$$

cannot be directly solved in a simple way as it was the one-dimensional equation. Nevertheless we can solve these equations by assuming certain particular expressions (with some undetermined coefficients) for the two functions x(t) and y(t).

(i) Bounded motions: Let us look for solutions with the following periodic form

$$x = A\sin(\omega t + \phi_1), \quad y = B\sin(\omega t + \phi_2), \tag{15}$$

where A, B, ϕ_1, ϕ_2 , and ω are real parameters. Then the equations (14) reduce to

$$AR\sin(\omega t + \phi_1) = 0$$
, $BR\sin(\omega t + \phi_2) = 0$,

where R is given by

$$R = \alpha^2 - \omega^2 - \lambda (A^2 + B^2) \omega^2 - \lambda^2 A^2 B^2 \omega^2 \sin^2 \phi_{12}, \quad \phi_{12} = \phi_1 - \phi_2.$$

Therefore the functions (15) are in fact solutions of (14) but with ω , that represents the angular frequency of the motion, λ -related with the coefficient α of the potential (that represents the frequency of the $\lambda = 0$ linear oscillator) by

$$\alpha^2 = M \omega^2$$
, $M = 1 + \lambda P_e$, $P_e = A^2 + B^2 + \lambda (A^2 B^2 \sin^2 \phi_{12})$.

Notice that the coefficient M is positive even for $\lambda = -|\lambda| < 0$ since in that case the amplitudes A and B must satisfy $A^2 + B^2 < 1/|\lambda|$.

Once we know the solution of the dynamics, we can obtain the constants values of the three integrals of motion, I_1 , I_2 , an J; they are given by

$$I_1 = (1 + \lambda B^2 \sin^2 \phi_{12}) A^2 \omega^2,$$

$$I_2 = (1 + \lambda A^2 \sin^2 \phi_{12}) B^2 \omega^2,$$

$$J = -\omega AB \sin \phi_{12}.$$

Using these expressions, we can obtain the values of the amplitudes A, B, as functions of the integrals of motion; we have

$$A = (\frac{1}{\omega})\sqrt{I_1 - \lambda J^2}, \quad B = (\frac{1}{\omega})\sqrt{I_2 - \lambda J^2},$$

so that the total energy becomes

$$E(\lambda) = \left(\frac{1}{2}\right) \left(A^2 + B^2\right) \omega^2 + \left(\frac{\lambda}{2}\right) \left(AB\sin\phi_{12}\right)^2 \omega^2,$$

$$= \left(\frac{\alpha^2}{2}\right) \left(\frac{P_e}{1 + \lambda P_e}\right) < \frac{\alpha^2}{2\lambda}.$$

Let us summarize. The four coefficients (A, B, ϕ_1, ϕ_2) remain arbitrary, the trajectories are sine-like periodic motions having the same frequency but (possibly) differing in amplitude and in phase, and the trajectories are "ellipses" in (x, y) plane. The situation is very similar to the one of the linear oscillator, the main difference laying in the frequency ω that is given by $\omega^2 = \alpha^2/M$ and it depends, therefore, on the position (initial data). We have two possibilities:

- If the parameter λ is negative $\lambda < 0$, then $\omega > \alpha$.
- If the parameter λ is positive $\lambda > 0$, then $\omega < \alpha$.

The energy can take any value for $\lambda < 0$, and it is always bounded by $E_{\alpha,\lambda} = (1/2)(\alpha^2/\lambda)$ for $\lambda > 0$, with the value of ω going down when the energy $E(\lambda)$ approaches to this upper value.

(ii) Unbounded motions: Let us analyze the solutions corresponding to $\lambda > 0$, and $E > E_{\alpha,\lambda}$.

If we assume the following expressions

$$x = A \sinh(\Omega t + \phi_1), \quad y = B \sinh(\Omega t + \phi_2),$$
 (16)

for x(t) and y(t), then we obtain that they are solutions of (14) with the condition that α and Ω must be λ -related by

$$\alpha^2 = M \Omega^2$$
, $M = -1 + \lambda P_h$, $P_h = A^2 + B^2 + \lambda (A^2 B^2 \sinh^2 \phi_{12})$.

The constant values of the three functions, I_1 , I_2 , an J, are given by

$$I_1 = (1 + \lambda B^2 \sinh^2 \phi_{12}) A^2 \Omega^2,$$

$$I_2 = (1 + \lambda A^2 \sinh^2 \phi_{12}) B^2 \Omega^2,$$

$$J = -\Omega AB \sinh \phi_{12},$$

and the coefficients A, B, can be rewritten as follows

$$A = (\frac{1}{\Omega})\sqrt{I_1 - \lambda J^2}, \quad B = (\frac{1}{\Omega})\sqrt{I_2 - \lambda J^2},$$

Finally, the total energy becomes

$$E(\lambda) = \left(\frac{1}{2}\right) \left(A^2 + B^2\right) \Omega^2 + \left(\frac{\lambda}{2}\right) \left(AB \sinh \phi_{12}\right)^2 \Omega^2$$
$$= \left(\frac{\alpha^2}{2}\right) \left(\frac{P_h}{\lambda P_h - 1}\right) > \frac{\alpha^2}{2\lambda}.$$

Figure III shows the form of the potential for $\lambda > 0$ and $\lambda < 0$.

(iii) Limiting unbounded motions: Let us analyze the very particular case characterized by $\lambda > 0$, and $E = E_{\alpha,\lambda}$ with $E_{\alpha,\lambda} = (1/2)(\alpha^2/\lambda)$.

If we assume the following expressions

$$x = A_1 t + B_1, \quad y = A_2 t + B_2,$$
 (17)

for the solutions, then we obtain that they are solutions of (14) with the following λ -dependent restriction for the four coefficients A_1 , A_2 , B_1 , and B_2 ,

$$\alpha^2 = \lambda P_L$$
, $P_L = A_1^2 + A_2^2 + \lambda (A_2 B_1 - A_1 B_2)^2$.

In this particular case the three functions, I_1 , I_2 , an J, take the form

$$I_{1} = A_{1}^{2} + \lambda (A_{2}B_{1} - A_{1}B_{2})^{2},$$

$$I_{2} = A_{2}^{2} + \lambda (A_{2}B_{1} - A_{1}B_{2})^{2},$$

$$J = A_{2}B_{1} - A_{1}B_{2},$$

and the coefficients A_1 , A_2 , can be rewritten as follows

$$A_1 = \sqrt{I_1 - \lambda J^2}$$
, $A_2 = \sqrt{I_2 - \lambda J^2}$.

Finally, the total energy becomes

$$E(\lambda) = (\frac{1}{2}) \left(A_1^2 + A_2^2 \right) + (\frac{\lambda}{2}) \left(A_2 B_1 - A_1 B_2 \right)^2 = \frac{\alpha^2}{2\lambda}.$$

Thus, the linear functions (17) appear, in this nonlinear system, as a border-line solution making separation between two different situations in the $\lambda > 0$ case: trigonometric periodic oscillations (15) for small energies and hyperbolic unbounded (scattering) evolutions (16) for high energies.

3.3 Hamiltonian approach

The Legendre transformation is given by

$$p_x = \frac{(1+\lambda y^2)v_x - \lambda xyv_y}{1+\lambda r^2}, \quad p_y = \frac{(1+\lambda x^2)v_y - \lambda xyv_x}{1+\lambda r^2},$$

so that the form of the angular momentum is preserved by the Legendre map, in the sense that we have $xp_y - yp_x = xv_y - yv_x$ (notice that this fact is consequence of the introduction of the term J in the definition of $T_2(\lambda)$), and the general expression for a λ -dependent Hamiltonian becomes

$$H(x, y, p_x, p_y; \lambda) = (\frac{1}{2}) \left[p_x^2 + p_y^2 + \lambda (xp_x + yp_y)^2 \right] + (\frac{1}{2}) \alpha^2 V(x, y)$$
 (18)

and hence the associated Hamilton-Jacobi equation takes the form

$$\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \lambda \left(x \frac{\partial S}{\partial x} + y \frac{\partial S}{\partial y}\right)^2 + \alpha^2 V(x, y) = 2E.$$

This equation is not separable in (x, y) coordinates because of the λ -dependent term; indeed the (x, y) coordinates are not even orthogonal. Nevertheless we will see that there exist three particular orthogonal coordinate systems, and three particular families of associated potentials, in which the Hamiltonian (18) admits Hamilton-Jacobi separability. To find them, we look for two one-parameter families of curves $f_1(x, y, c_1) = 0$ and $f_2(x, y, c_2) = 0$, $g(\lambda)$ -orthogonal to the level set curves of x and y, we obtain respectively,

$$x = c_1 \sqrt{1 + \lambda y^2}$$
, and $y = c_2 \sqrt{1 + \lambda x^2}$.

These two expressions suggest us to consider two particular systems, that we will denote by (z_x, y) and (x, z_y) , that can be seen as λ -dependent deformations of the (x, y) coordinates; the third system is just the polar coordinate system.

(i) In terms of the new coordinates

$$(z_x, y), \quad z_x = \frac{x}{\sqrt{1 + \lambda y^2}},$$

the Hamilton-Jacobi equation becomes

$$(1 + \lambda z_x^2) \left(\frac{\partial S}{\partial z_x}\right)^2 + (1 + \lambda y^2)^2 \left(\frac{\partial S}{\partial y}\right)^2 + \alpha^2 (1 + \lambda y^2) V = 2(1 + \lambda y^2) E$$

so if the potential V(x,y) can be written on the form

$$V = \frac{W_1(z_x)}{1 + \lambda y^2} + W_2(y) \tag{19}$$

then the equation becomes separable. The potential is therefore integrable with the following two quadratic integrals of motion

$$I_{1}(\lambda) = (1 + \lambda r^{2})p_{x}^{2} + \alpha^{2}W_{1}(z_{x}),$$

$$I_{2}(\lambda) = (1 + \lambda r^{2})p_{y}^{2} - \lambda J^{2} + \alpha^{2}(W_{2}(y) - \frac{\lambda y^{2}}{1 + \lambda y^{2}}W_{1}(z_{x})).$$

(ii) In terms of

$$(x, z_y), \quad z_x = \frac{y}{\sqrt{1 + \lambda r^2}},$$

which is the symmetric one of the previous change (i), the Hamilton-Jacobi equation becomes

$$(1+\lambda x^2)^2 \left(\frac{\partial S}{\partial x}\right)^2 + (1+\lambda z_y^2) \left(\frac{\partial S}{\partial z_y}\right)^2 + \alpha^2 (1+\lambda x^2) V = 2(1+\lambda x^2) E,$$

and, therefore, if the potential V(x,y) can be written on the form

$$V = W_1(x) + \frac{W_2(z_y)}{1 + \lambda x^2} \tag{20}$$

then the equation becomes separable. The potential is therefore integrable with the following two quadratic integrals of motion

$$I_1(\lambda) = (1 + \lambda r^2)p_x^2 - \lambda J^2 + \alpha^2 \left(W_1(x) - \frac{\lambda x^2}{1 + \lambda x^2}W_1(z_y)\right),$$

$$I_2(\lambda) = (1 + \lambda r^2)p_y^2 + \alpha^2 W_2(z_y).$$

(iii) Finally we may use polar coordinates (r, ϕ) . Here the λ -dependent Hamiltonian (18) is given by

$$H(r, \phi, p_r, p_{\phi}; \lambda) = (\frac{1}{2}) \left[(1 + \lambda r^2) p_r^2 + \frac{p_{\phi}^2}{r^2} \right] + (\frac{\alpha^2}{2}) V(r, \phi)$$

so that the Hamilton-Jacobi equation is given by

$$(1+\lambda\,r^2) \Big(\frac{\partial S}{\partial r}\Big)^2 + \frac{1}{r^2} \Big(\frac{\partial S}{\partial \phi}\Big)^2 + \alpha^2\,V(r,\phi) = 2E\,.$$

Let us suppose that the potential V takes the form

$$V = F(r) + \frac{G(\phi)}{r^2}, \qquad (21)$$

then the equation admits separability

$$\left[r^2(1+\lambda r^2)\left(\frac{\partial S}{\partial r}\right)^2 + r^2(\alpha^2 F(r) - 2E)\right] + \left[\left(\frac{\partial S}{\partial \phi}\right)^2 + \alpha^2 G(\phi)\right] = 0.$$

The potential V is integrable with the following two quadratic integrals of motion

$$I_{1}(\lambda) = (1 + \lambda r^{2}) p_{r}^{2} + \left(\frac{1 - r^{2}}{r^{2}}\right) p_{\phi}^{2} + \alpha^{2} \left[F(r) + \left(\frac{1 - r^{2}}{r^{2}}\right) G(\phi)\right],$$

$$I_{2}(\lambda) = p_{\phi}^{2} + \alpha^{2} G(\phi).$$

It is clear that f G=0 then V is a λ -dependent central potential and the function I_2 just becomes $I_2=p_\phi^2$.

In these three separable cases the potential $V(\lambda)$ is integrable with two quadratic constants of motion, $I_1(\lambda)$ and $I_2(\lambda)$, in such a way that

$$H(\lambda) = \left(\frac{1}{2}\right) \left(I_1(\lambda) + I_2(\lambda)\right)$$

That is, there exist three different ways in which the Hamiltonian $H(\lambda)$ admits a decomposition as sum of two integrals; notice that in the linear case $\lambda = 0$ they reduce to only two since the two first cases, (i) and (ii), coincide.

We recall that a potential V is called super-separable if it is separable in more than one system of coordinates [25]-[30]. The potential

$$V_2(\lambda) = \frac{\alpha^2}{2} \left(\frac{x^2 + y^2}{1 + \lambda (x^2 + y^2)} \right)$$

can be alternatively written as follows

$$V_{2}(\lambda) = \frac{\alpha^{2}}{2} \left(\frac{1}{1 + \lambda y^{2}} \right) \left[\frac{z_{x}^{2}}{1 + \lambda z_{x}^{2}} + y^{2} \right]$$

$$= \frac{\alpha^{2}}{2} \left(\frac{1}{1 + \lambda x^{2}} \right) \left[x^{2} + \frac{z_{y}^{2}}{1 + \lambda z_{y}^{2}} \right]$$

$$= \frac{\alpha^{2}}{2} \left(\frac{r^{2}}{1 + \lambda r^{2}} \right).$$

Therefore, it is super-separable since it is separable in three different systems of coordinates $(z_x, y), (x, z_y)$, and (r, ϕ) . Because of this the Hamiltonian

$$H = \left(\frac{1}{2}\right) \left[p_x^2 + p_y^2 + \lambda \left(x p_x + y p_y \right)^2 \right] + \frac{\alpha^2}{2} \left(\frac{x^2 + y^2}{1 + \lambda \left(x^2 + y^2 \right)} \right)$$
 (22)

admits the following decomposition

$$H = H_1 + H_2 - \lambda H_3$$

where the three partial functions H_1 , H_2 , and H_3 , are given by

$$H_{1} = \frac{1}{2} \left[(1 + \lambda r^{2}) p_{x}^{2} + \alpha^{2} \left(\frac{x^{2}}{1 + \lambda r^{2}} \right) \right],$$

$$H_{2} = \frac{1}{2} \left[(1 + \lambda r^{2}) p_{y}^{2} + \alpha^{2} \left(\frac{y^{2}}{1 + \lambda r^{2}} \right) \right],$$

$$H_{3} = \frac{1}{2} (x p_{y} - y p_{x})^{2},$$

each one of these three terms has a vanishing Poisson bracket with H for any value of the parameter λ

$$\{H, H_1\} = 0, \quad \{H, H_2\} = 0, \quad \{H, H_3\} = 0.$$

So the total Hamiltonian can be written as a sum, not of two, but of three integrals of motion. The third one, which represents the contribution of the angular momentum J to H, has the parameter λ as coefficient; so it vanishes in the linear limit $\lambda \to 0$.

4 n-dimensional Oscillator

We have studied with a great detail the λ -dependent nonlinear bi-dimensional oscillator; nevertheless, it is clear that this particular Lagrangian (Hamiltonian) system admits a direct extension to n dimensions.

4.1 Lagrangian formalism

The n-dimensional λ -dependent Lagrangian is given by

$$L(\lambda) = \frac{1}{2} \left(\frac{1}{1 + \lambda r^2} \right) \left[\sum_{i} v_i^2 + \lambda \sum_{i < j} J_{ij}^2 \right] - \frac{\alpha^2}{2} \left(\frac{r^2}{1 + \lambda r^2} \right)$$
 (23)

where we have made use of the notation

$$r^2 = \sum_i x_i^2$$
, and $J_{ij} = x_i v_j - x_j v_i$, $i, j = 1, ..., n$.

The λ -dependent Euler-Lagrange vector field Γ_{λ} takes the form

$$\Gamma_{\lambda} = v_k \frac{\partial}{\partial x_k} + F_k(x, v, \lambda) \frac{\partial}{\partial v_k}$$

where the *n* functions $F_k = F_k(x, v, \lambda)$, k = 1, ..., n, are given by

$$F_k = -\alpha^2 \left(\frac{x_k}{1+\lambda r^2}\right) + \lambda \left[\sum_i v_i^2 + \lambda \sum_{i < j} J_{ij}^2\right] \left(\frac{x_k}{1+\lambda r^2}\right).$$

Same as in the n=2 case, we can obtain the explicit expressions for the solutions of the dynamics. We also have, in this n-dimensional case, a quasi-harmonic oscillatory motion for $\lambda < 0$, and two different qualitative behaviours with a border-case when $\lambda > 0$.

(i) Periodic motions: We can assume that the solutions $x_i = x_i(t)$ of the n equations

$$\ddot{x}_i = F_i(x, \dot{x}; \lambda), \qquad i = 1, \dots, n,$$

are periodic functions of the form

$$x_i = A_i \sin(\omega t + \phi_i), \qquad i = 1, \dots, n,$$

Then we arrive at

$$A_i R \sin(\omega t + \phi_i) = 0, \qquad i = 1, \dots, n,$$

where R is given by

$$R = \alpha^2 - \left(1 + \lambda \sum_i A_i^2\right) \omega^2 - \lambda^2 \left(\sum_{i,j} A_i^2 A_j^2 \sin^2 \phi_{ij}\right) \omega^2.$$

Therefore, we have $\omega^2 \neq \alpha^2$ but $\omega^2 = \alpha^2/M$ with $M = M(\lambda)$ given by

$$M(\lambda) = 1 + \lambda P_e^{(n)}, \quad P_e^{(n)} = \sum_i A_i^2 + \lambda \left(\sum_{i,j} A_i^2 A_j^2 \sin^2 \phi_{ij} \right),$$

and the energy $E(\lambda)$ is given by

$$E(\lambda) = \left(\frac{1}{2}\right) \left(\sum_{i} A_{i}^{2}\right) \omega^{2} + \left(\frac{\lambda}{2}\right) \sum_{i,j} \left(A_{i} A_{j} \sin \phi_{ij}\right)^{2} \omega^{2},$$
$$= \left(\frac{\alpha^{2}}{2}\right) \left(\frac{P_{e}^{(n)}}{1 + \lambda P_{e}^{(n)}}\right).$$

(ii) Unbounded motions: If we assume that the solutions $x_i = x_i(t)$ are of the form

$$x_i = A_i \sinh(\Omega t + \phi_i), \qquad i = 1, \dots, n,$$

then we arrive to $\Omega^2 = \alpha^2/M$ with $M(\lambda)$ given by

$$M(\lambda) = -1 + \lambda P_h^{(n)}, \quad P_h^{(n)} = \sum_i A_i^2 + \lambda \left(\sum_{i,j} A_i^2 A_j^2 \sinh^2 \phi_{ij} \right),$$

and the energy $E(\lambda)$ is given by

$$E(\lambda) = \left(\frac{1}{2}\right) \left(\sum_{i} A_{i}^{2}\right) \Omega^{2} + \left(\frac{\lambda}{2}\right) \sum_{i,j} \left(A_{i} A_{j} \sin \phi_{ij}\right)^{2} \Omega^{2}$$
$$= \left(\frac{\alpha^{2}}{2}\right) \left(\frac{P_{h}^{(n)}}{\lambda P_{h}^{(n)} - 1}\right).$$

(iii) Finally, in the positive $\lambda > 0$ case, there exists a very particular border-case that makes separation between the behaviours (i) and (ii). It is characterized by the a value of the energy E given by $E = E_{\alpha,\lambda}$, and its time-evolution is represented by linear functions

$$x_i = A_i t + B_i$$
, $i = 1, \ldots, n$,

with the following λ -dependent restriction for the 2n coefficients A_i , B_i ,

$$\alpha^2 = \lambda P_L, \quad P_L^{(n)} = \sum_i A_i^2 + \lambda \sum_{i,j} (A_i B_j - A_j B_i)^2.$$

Concerning the energy $E(\lambda)$ it takes the particular value

$$E(\lambda) = (\frac{1}{2}) \sum_{i} A_i^2 + (\frac{\lambda}{2}) \sum_{i,j} (A_i B_j - A_j B_i)^2 = \frac{\alpha^2}{2 \lambda}.$$

4.2 Hamiltonian formalism

The *n*-dimensional λ -deformed Hamiltonian is given by

$$H = \frac{1}{2} \left[\sum_{i} p_i^2 + \lambda \left(\sum_{i} x_i p_i \right)^2 \right] + \frac{\alpha^2}{2} \left(\frac{r^2}{1 + \lambda r^2} \right), \quad r^2 = \sum_{i} x_i^2.$$
 (24)

It can be rewritten as a sum of N=(1/2)n(n+1) quadratic terms as follows

$$H = \left(\frac{1}{2}\right) \left(\sum_{k} I_k(\lambda)\right) - \left(\frac{\lambda}{2}\right) \left(\sum_{i < j} J_{ij}^2\right)$$

where the functions $I_k(\lambda)$ and J_{ij} , which are given by

$$I_k(\lambda) = (1 + \lambda r^2) p_k^2 + \alpha^2 \left(\frac{x_k^2}{1 + \lambda r^2}\right), \quad k = 1, \dots, n,$$

$$J_{ij} = x_i p_j - x_j p_i, \quad i, j = 1, \dots, n,$$

are constants of motion

$$\{H, I_k(\lambda)\} = 0, \qquad \{H, J_{ij}\} = 0.$$

Moreover, the λ -dependent functions

$$I_{ij}(\lambda) = (1 + \lambda r^2) p_i p_j + \alpha^2 \left(\frac{x_i x_j}{1 + \lambda r^2}\right), \quad i, j = 1, \dots, n,$$
 (25)

are constants of motion as well. Of course, the existence of these three different families, $I_k(\lambda)$, J_{ij} , and $I_{ij}(\lambda)$, means that all these integrals cannot be independent since the maximum number of (time-independent) functionally independent integrals is N = 2n - 1. In order to obtain a fundamental set of independent integrals we can choose the n functions $I_k(\lambda)$ and n-1 of the angular momenta; an example is given by

$$(I_k(\lambda), J_{i,i+1}), \quad k = 1, \dots, n, \quad i = 1, \dots, n-1.$$

In fact this situation is rather the same that one finds in the linear $\lambda = 0$ case but with two important distinctions: first that all these facts remain valid also for the unbounded (or scattering) motions, present when $\lambda > 0$ and second that the algebra of Poisson brackets seems to be quadratic.

We close this section observing that, if we call super-separable to a system that admits Hamilton-Jacobi separation of variables (Schroedinger in the quantum case) in more than one coordinate system, then quadratic super-integrability (i.e., super-integrability with linear or quadratic constants of motion) can be considered as a property arising from super-separability. In this λ -dependent case, the three families of constants of motion are of such a class (J_{ij} are linear and $I_k(\lambda)$ and $I_{ij}(\lambda)$ are quadratic), so we can conclude that the super-integrability of the Hamiltonian (24) arises from its multiple separability. In the linear $\lambda = 0$ case, since the Hamiltonian is directly separable in the n-dimensional cartesian system, the Hamiltonian is just the sum of the n partial one-dimensional energies; in the general non-linear $\lambda \neq 0$ case, the functions $I_i(\lambda)$, and J_{ij} $i, j = 1, \ldots, n$, arise from separability in the n-dimensional versions of the two-dimensional coordinates (z_x, y) and (x, z_y) studied in Sec. 3 and in the n-dimensional spherical system. Finally, there exist many different sets of n commuting constants; as an example, in the n = 3 case we have the following three sets of involutive integrals

$$\begin{array}{l} \left(\,I_{1}\,,\,I_{2}-\lambda\,J_{12}^{2}\,,\,I_{3}-\lambda\,(J_{23}^{2}+J_{31}^{2})\,\right),\\ \left(\,I_{1}-\lambda\,(J_{12}^{2}+J_{31}^{2})\,,\,I_{2}\,,\,I_{3}-\lambda\,J_{23}^{2}\,\right),\\ \left(\,I_{1}-\lambda\,J_{31}^{2}\,,\,I_{2}-\lambda\,(J_{12}^{2}+J_{23}^{2})\,,\,I_{3}\,\right), \end{array}$$

as well as $(I_1 + I_2 + I_3, J_{ij}, J^2)$. It is clear from this example that in the general *n*-dimensional case there exist many more different ways of constructing involutive sets of *n* integrals.

5 A Geometric Interpretation

In this section we will discuss some additional properties that will prove to be related with a new geometric interpretation. In particular, we will see that this quasi-harmonic nonlinear oscillator turns out to be closely related with the harmonic oscillator on a space of constant curvature.

5.1 Some additional properties

We begin by considering two remarkable properties; the first one is concerned with the symmetries of two-dimensional system and the second one with the Lagrangian of the one-dimensional oscillator.

(i) The three λ -dependent vector fields X_1, X_2, X_J , obtained in Sec. 2 as the Killing vectors of the metric ds^2 or, equivalently, as the Noether symmetries of the Lagrangian $L(\lambda) = T_2(\lambda)$ of the λ -dependent free particle, close the following Lie algebra,

$$[X_1(\lambda), X_2(\lambda)] = \lambda X_J, \quad [X_1(\lambda), X_J] = X_2(\lambda), \quad [X_2(\lambda), X_J] = -X_1(\lambda).$$
 (26)

This means that the 2-d configuration space has a three-dimensional symmetry Lie algebra, hence a maximal one, and this implies that the space should be of constant curvature. Indeed the Lie algebra (26) is isomorphic to the Euclidean algebra in the particular $\lambda=0$ case, and to the Lie algebra of the isometries of the two-dimensional spherical ($\lambda<0$) and hyperbolic spaces ($\lambda>0$) in the general $\lambda\neq 0$ case.

It seems, therefore, that there exists a certain relation between this λ -dependent nonlinear oscillator and the properties of the two-dimensional spaces of constant curvature. At this point we recall that we have proved in Sec. 4 that this nonlinear system is well defined for any number n of degrees of freedom; therefore, it is natural to guess that if such relationship exists then it must be true, not only for n = 2, but for any dimension.

Although the n=1 case can be considered as a very special case, it seems convenient to go back to the one-dimensional oscillator and analyze again its properties but now in relation with the above point (i). We have obtained the following property that concerns the one-dimensional nonlinear system.

(ii) Let us consider the change $(x, v_x) \to (q, v_q)$ given by

$$q = \left(\frac{1}{\sqrt{\lambda}}\right) \sinh^{-1}(\sqrt{\lambda}x), \quad \lambda > 0,$$

then the Lagrangian

$$L(x, v_x; \lambda) = \frac{1}{2} \left(\frac{1}{1 + \lambda x^2} \right) (v_x^2 - \alpha^2 x^2)$$
 (27)

becomes

$$L(q, v_q; \lambda) = \frac{1}{2} v_q^2 - \left(\frac{\alpha^2}{2\lambda}\right) \tanh^2(\sqrt{\lambda} q).$$
 (28)

In the negative case, $\lambda < 0$, $\lambda = -|\lambda|$, the corresponding change is given by

$$q = \left(\frac{1}{\sqrt{|\lambda|}}\right) \sin^{-1}(\sqrt{|\lambda|} x), \quad \lambda = -|\lambda|,$$

and then we arrive at

$$L(q, v_q; \lambda) = \frac{1}{2} v_q^2 - \left(\frac{\alpha^2}{2|\lambda|}\right) \tan^2(\sqrt{|\lambda|} q).$$
 (29)

Hence, we can remove the λ parameter from the kinetic energy $T_1(\lambda)$, but the price for this simplification is that the potential V drops its rational character and becomes a trigonometric or hyperbolic squared tangent function

$$V(q;\lambda) = \frac{1}{2} \left(\frac{\alpha^2}{\lambda}\right) \tanh^2(\sqrt{\lambda} q), \quad \text{for} \quad \lambda > 0,$$

$$V(q;\lambda) = \frac{1}{2} \left(\frac{\alpha^2}{|\lambda|}\right) \tan^2(\sqrt{|\lambda|} q), \quad \text{for} \quad \lambda < 0.$$

From a practical point of view, this change does not simplify really the problem, since the two new equations

$$\sqrt{|\lambda|} \ddot{q} + \alpha^2 F(q; \lambda) = 0,$$

with $F = F(q, \lambda)$ given by

$$F = \frac{\sinh\left(\sqrt{\lambda}\,q\right)}{\cosh^3(\sqrt{\lambda}\,q)}, \quad \lambda > 0, \quad \text{and} \quad F = \frac{\sin\left(\sqrt{|\lambda|}\,q\right)}{\cos^3(\sqrt{|\lambda|}\,q)}, \quad \lambda < 0,$$

are not easier of solving than the original one (1). Nevertheless, from a more qualitative viewpoint, the new aspect adopted by this one-dimensional Lagrangian is a very interesting fact since the form obtained for the new potential $V(q, \lambda)$ is closely related with the polar coordinate (ρ, ϕ) expression for the two-dimensional potential of the harmonic oscillator on two-dimensional spaces of constant curvature, previously studied (by two of the present authors) in Refs. [16]-[18].

It is clear from properties (i) and (ii) that we can conjecture the existence of a direct relationship between this particular nonlinear oscillator and the harmonic oscillator on spaces of constant curvature. Another remarkable property is that the set of integrals of motion (25) is analogous to the ones appearing in the n-dimensional version of the Smorodinsky-Winternitz system in curved spaces [31].

5.2 The harmonic oscillator on spaces of constant curvature

Higgs [32] and Leemon [33] analyzed the characteristics of the two fundamental central potentials, Kepler problem and harmonic oscillator, on the N-dimensional sphere. Since then a certain number of authors have studied this question from both the classical and the quantum points of view [34]-[40]. Next we recall some of the basic properties of the formalism studied in [16]-[18].

We begin for the following three basic ideas:

• The harmonic oscillator is a system that is well defined in all the three two-dimensional spaces of constant curvature (sphere S^2 , Euclidean plane \mathbb{E}^2 , and hyperbolic plane H^2).

- A joint approach, where the usual Euclidean system and the two curved systems (defined on S^2 and H^2) can be studied, all the three, at the same time, is possible using the curvature κ as a parameter.
- Thus the spherical and hyperbolic oscillators can be considered as *curvature deformations* of the well known "flat" Euclidean oscillator which arises as a very particular case of the more general "curved" systems.

This two last points mean that all the theory must be developed by making use of κ -dependent mathematical expressions leading to general κ -dependent properties. The specific properties characterizing the harmonic oscillator on the sphere, on the Euclidean plane, or on the hyperbolic plane, are then obtained particularizing for $\kappa > 0$, $\kappa = 0$, or $\kappa < 0$. In order to present these expressions in a form which holds simultaneously for any value of κ , the theory can be developed by making use of the following "tagged" trigonometric functions

$$C_{\kappa}(x) = \begin{cases} \cos\sqrt{\kappa} x & \text{if } \kappa > 0, \\ 1 & \text{if } \kappa = 0, \\ \cosh\sqrt{-\kappa} x & \text{if } \kappa < 0, \end{cases} \qquad S_{\kappa}(x) = \begin{cases} \frac{1}{\sqrt{\kappa}} \sin\sqrt{\kappa} x & \text{if } \kappa > 0, \\ x & \text{if } \kappa = 0, \\ \frac{1}{\sqrt{-\kappa}} \sinh\sqrt{-\kappa} x & \text{if } \kappa < 0, \end{cases}$$
(30)

If we make use of this κ -dependent notation in the polar coordinates (ρ, ϕ) system, then the differential element of distance, on the three spaces (S^2, \mathbb{E}^2, H^2) , can be written as follows

$$ds_{\kappa}^2 = d\rho^2 + S_{\kappa}^2(\rho) d\phi^2,$$

It reduces to

$$ds_1^2 = d\rho^2 + (\sin^2 \rho) d\phi^2$$
, $ds_0^2 = d\rho^2 + \rho^2 d\phi^2$, $ds_{-1}^2 = d\rho^2 + (\sinh^2 \rho) d\phi^2$,

in the three particular cases of the unit sphere, Euclidean plane, and 'unit' Lobachewski plane. Consequently, the Lagrangian for the geodesic (free) motion on the spaces (S^2, \mathbb{E}^2, H^2) is given by the kinetic term arising from the metric

$$L(\kappa) = T(\kappa) = \left(\frac{1}{2}\right) \left(v_{\rho}^2 + S_{\kappa}^2(\rho)v_{\phi}^2\right),\,$$

and the Lagrangian for a general mechanical system (Riemmanian metric minus a potential) has the following form

$$L(\kappa) = \left(\frac{1}{2}\right) \left(v_{\rho}^2 + S_{\kappa}^2(\rho)v_{\phi}^2\right) - U(\rho, \phi, \kappa).$$

It is clear that the well known expression for a natural Lagrangian on the Euclidean plane

$$L = (\frac{1}{2})(v_{\rho}^2 + \rho^2 v_{\phi}^2) - V(\rho, \phi), \quad V(\rho, \phi) = U(\rho, \phi, 0),$$

is obtained as the particular $\kappa = 0$ case of $L(\kappa)$.

Until now we have considered general aspects of the theory of Lagrangian systems on curved spaces. Now let us turn our attention to the spherical and hyperbolic harmonic oscillators; it is characterized by the following Lagrangian with curvature κ

$$L = (\frac{1}{2}) \left(v_{\rho}^2 + S_{\kappa}^2(\rho) v_{\phi}^2 \right) - (\frac{1}{2}) \omega_0^2 T_{\kappa}^2(\rho).$$

where the κ -dependent tangent $T_{\kappa}(\rho)$ is defined in the natural way $T_{\kappa}(\rho) = S_{\kappa}(\rho)/C_{\kappa}(\rho)$. In this way, the harmonic oscillator on the unit sphere (Higgs oscillator), on the Euclidean plane, or on the unit Lobachewski plane, arise as the following three particular cases

$$U_1(\rho) = (\frac{1}{2}) \,\omega_0^2 \,\tan^2 \rho \,, \quad V(\rho) = U_0(\rho) = (\frac{1}{2}) \,\omega_0^2 \,\rho^2 \,, \quad U_{-1}(\rho) = (\frac{1}{2}) \,\omega_0^2 \,\tanh^2 \rho \,.$$

The Euclidean oscillator $V(\rho) = U_0(\rho)$ (parabolic potential without singularities) appears in this formalism as making a separation between two different situations. The spherical potential is represented by a well with singularities on the border (impenetrable walls at the equatorial circle $\rho = \pi/2\sqrt{\kappa}$ if the potential center is placed at the poles), and the hyperbolic potential by a well with finite depth since for $\kappa < 0$, $\kappa = -|\kappa|$, we have $U_{\kappa}(\rho) \rightarrow (1/2)(\omega_0^2/|\kappa|)$ when $\rho \rightarrow \infty$. Actually, the Scarf potential $V(x) = \gamma/\cos^2(x)$, which differs in a constant term from $U_1(x)$, has been studied in solid state physics and has many interesting properties [41]-[43]. Figure IV plots $U_{\kappa}(\rho)$, for the three particular cases of the unit sphere, Euclidean plane, and 'unit' Lobachewski plane; notice the great resemblance with Figure III.

5.3 On the existence of a relation between two dynamics

The abovementioned property (ii), for the n=1 case, suggests that the Lagrangian (27) and the equation (1) is a nonlinear model for an harmonic oscillator in the circle S^1 ($\lambda < 0$) or in the hyperbolic line ($\lambda > 0$). In a similar way, if we consider the two properties (i) and (ii) together, we arrive at the conclusion that this correspondence must also exist for the nonlinear n=2 oscillator. Remark that in the n=1 case, S^1 and H^1 must be understood as one-dimensional spaces obtained by endowing each single geodesic of S^2 or H^2 with the induced metric. Motion on S^1 and H^1 will correspond to the J=0 radial motions on S^2 or H^2 .

Hence, the Lagrangian (27) for the equation (1) can be considered as a nonlinear model on the \mathbb{R} -line for the harmonic oscillator on the circle S^1 and the hyperbolic line H^1 ; the Lagrangian $L = T_2(\lambda) - V_2(\lambda)$ of equation (13) as a nonlinear \mathbb{R}^2 -model for the harmonic oscillator on the sphere S^2 and the hyperbolic plane H^2 ; finally, the more general n-dimensional Hamiltonian given by equation (24) as a nonlinear \mathbb{R}^n -model for the harmonic oscillator on the n-dimensional spaces S^n and H^n . So we arrive to the conclusion that an harmonic oscillator, that is a linear system, when is defined on a space of constant curvature turns out to be equivalent to a nonlinear oscillator on \mathbb{R} , \mathbb{R}^2 or \mathbb{R}^n , with the nonlinear parameter λ playing the role of the (negative of the) curvature κ . Notice that, in dynamical terms, this equivalence is a relation of conjugacy, and also that the existence of this relation can be

considered as the origin of the quasi-harmonic behaviour obtained for the solutions of the nonlinear system.

The existence of this relationship makes this nonlinear \mathbb{R}^2 (or \mathbb{R}^n) system even more interesting. From an abstract or qualitative viewpoint, the system on the sphere S^2 (or on the hyperbolic space H^2) may be considered as the more fundamental one; nevertheless, we have been able to solve the equations and to obtain the explicit solution for the dynamics because we were working with the nonlinear λ -dependent \mathbb{R}^2 -model (or \mathbb{R}^n -model). Hence, the nonlinear system studied in this article results to be much more appropriate for the explicit resolution of problems.

6 Final Comments and Outlook

We have proved that the Lagrangian (13) is the appropriate two-dimensional version of the Lagrangian (2), (27), and we have solved the nonlinear equations (14) in the two cases $\lambda < 0$ and $\lambda > 0$. Moreover we have proved in Sec. 4 that this nonlinear system admits a n-dimensional version given by (23) and (24), and we have also solved the correspondent system of n equations.

We think that all these results suggest the study of some related questions among them we point out the following: Firstly, the Poisson brackets of the integrals of motion for the 2-dimensional Hamiltonian (22) or the n-dimensional Hamiltonian (24) seems to close a quadratic Poisson algebra (see e.g. Ref. [44]); this is a very interesting problem that deserves to be studied. Secondly, the geometric relations obtained in the last Sec. 5, is a matter related with the existence of two different but conjugate dynamical systems. Conjugate systems are systems related by diffeomorphisms preserving the fundamental properties; so it is convenient to develop a deeper analysis of this particular relation of conjugacy existing between the nonlinear λ -dependent oscillator and the harmonic oscillator on the sphere S^2 or the hyperbolic plane H^2 . Thirdly, it is clear that the calculus are easier of handle when working with the λ -dependent formalism than when working on the constant curvature spaces; therefore, when looking for new dynamical results on the spaces such as S^2 or H^2 . a practical strategy will be to first consider the question by using Lagrangians such as (13) or (23). Fourthly, in Refs. [17] and [18] we have studied not only the central 1:1 oscillator given by the function $V_{\kappa}(\rho) = (1/2) \omega_0^2 T_{\kappa}^2(\rho)$ but also some others non-central non-isotropic oscillators $V_{\kappa}(\rho,\phi)$; it will be convenient to consider these non-central problems also inside this λ -dependent formalism. Finally, we mention the study of the quantized versions of all these nonlinear systems. Notice that, from the quantum point of view, as it is an oscillator with a position-dependent effective mass, there is a problem with the appropriate order for the factors in the kinetic term. We think that all these problems are examples of some open questions that must be investigated.

Acknowledgments.

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Figure Captions

FIGURE I. Plot of $V(\lambda) = (1/2) (\alpha^2 x^2)/(1 + \lambda x^2)$, $\alpha = 1$, $\lambda < 0$, as a function of x, for $\lambda = -2$ (upper curve), and $\lambda = -1$ (lower curve).

FIGURE II. Plot of $V(\lambda) = (1/2) (\alpha^2 x^2)/(1 + \lambda x^2)$, $\alpha = 1$, $\lambda > 0$, as a function of x, for $\lambda = 1$ (upper curve), and $\lambda = 2$ (lower curve).

FIGURE III. Plot of $V_2(\lambda) = (1/2) (\alpha^2 r^2)/(1 + \lambda r^2)$, $\alpha = 1$, as a function of r, for $\lambda = -1$ (upper curve), $\lambda = 0$ (dashed line) and $\lambda = 1$ (lower curve).

FIGURE IV. Plot of $U_{\kappa}(\rho) = (1/2) \omega_0^2 T_{\kappa}^2(\rho)$, $\omega_0 = 1$, as a function of ρ , for $\kappa = -1$ (lower curve), $\kappa = 0$ (dash line), and $\kappa = 1$ (upper curve).

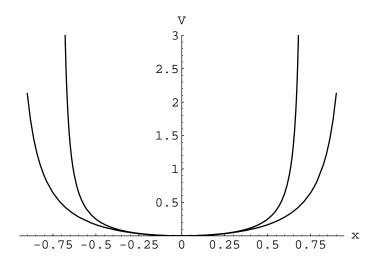


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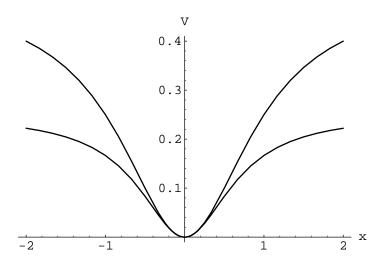


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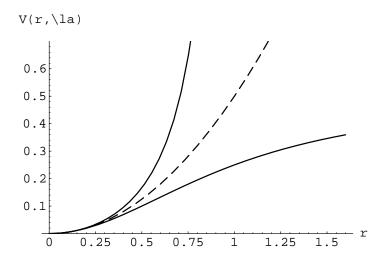


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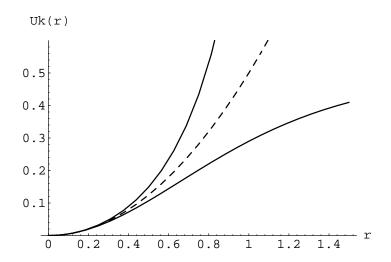


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